BACKLASH IDENTIFICATION: A TWO STEP APPROACH

Laleh Ravanbod-Shirazi * **Alina Besançon-Voda** **,[1]

* *shirazi@lag.ensieg.inpg.fr*
** *Alina.Besançon@inpg.fr*
Laboratoire d'Automatique de Grenoble
ENSIEG, B.P. 46, 38402 Saint Martin d'Here, FRANCE

Abstract: In this paper, a two step approach for backlash identification is proposed. In the first step, using a physical relation between backlash amplitude, the vibration produced by backlash and the backlash input, a pre-estimation is established. In the second step, a least square criterion is minimized around the pre-estimation. The method is tested in simulation on a system consisting of a driving motor and a load coupled to it through a gear and a shaft. The method is simple and can be easily generalized to systems containing many shafts. The simulation results confirm the good performance of the method.

Keywords: Backlash, Identification, Oscillation, Nonlinear model, Regression

1. INTRODUCTION

Backlash is one of the most important non-linearity that limits the performance of speed and position control. In industrial drives, backlash is present in elements as gear boxes and flexible couplings. It is a destablizing factor for example, if the gear backlash is hold in motor drive systems with a torsional loads, the results of a speed step response is with extended duration of vibration. In the literature, several methods for backlash amplitude identification, modeled by dead zone, have been studied. In most of these works, the system models are (see figure 1): Hammerstein (Tao and Kokotovic, 1993), Wiener (Tian and Tao, 1997); (Woo *et al.*, 1998) or Hammerstein-Wiener (Tao, 1996). The common point of all these approaches is that the system is divided in non linear blocks (backlash), and one or several linear blocks representing the other parts of the system. Unfortunately, these methods cannot be applied to a system with feedbacks (figure 2). Other researchers (Stein and Wang, 1998) have tried to estimate the backlash amplitude, θ, using the vibrations (amplitudes) of the motor speed, ω_M, or of the load speed, ω_L, (figure 3). The phenomenon of vibration occurs when the input

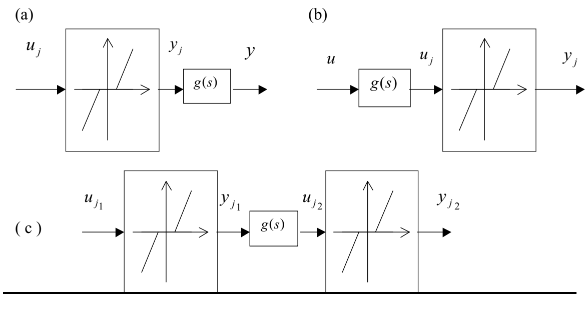

Fig. 1. The models used for backlash identification: a) Hammerstein, b) Wiener, c) Hammerstein-Wiener.

[1] The authors would like to thank PERT of ALSTOM Technologies for the useful discussions on the industrial interest of this study

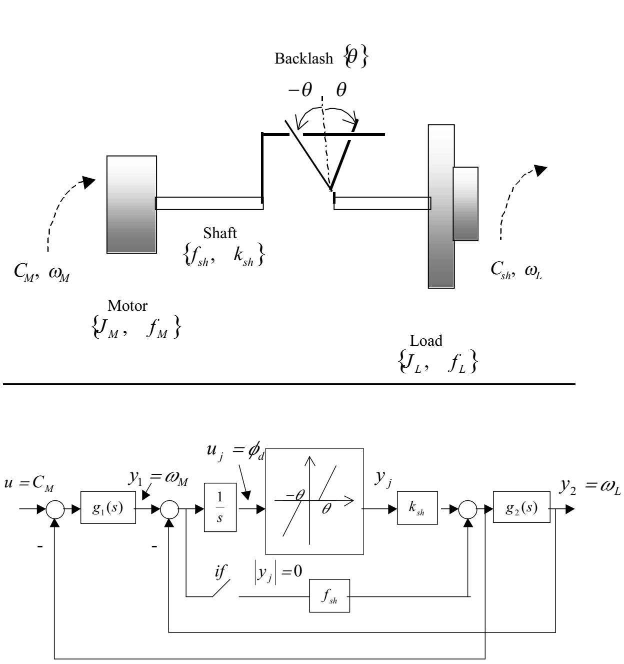

Fig. 2. The motor drive system with a torsional load and the bloc diagram representation.

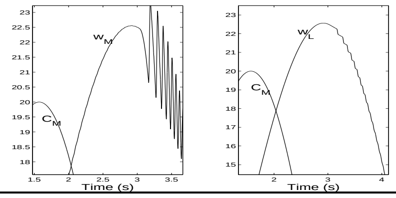

Fig. 3. Vibration introduced by a sinus input.

torque, C_M, excites the backlash. It means that an input, who changes the rotation direction of the motor. However, as specify the authors, the method is sensible to the calibration of the system. Moreover, the generalization of the approach to the systems containing several shafts is not obvious.

In this paper, we present an original two steps approach. In the first step, a pre-estimation of the backlash amplitude (θ in figure 2) is performed based on the physical relation between the vibration instants, backlash amplitude and the backlash input. In the second step, to obtain the final estimation of the backlash amplitude, a least square criteria is minimized around the pre-estimation. The identification signals are the noisy measurements of the motor speed, ω_{M_n}, and load speed, ω_{L_n}. [2] It is assumed that the other parameters (see figure 2), are random variables centered around their exact values.

The simulation results confirm the good performance of the proposed method and show that the pre-estimation is crucial for obtaining a good final estimation using the least square approach.

For a system composed of many shafts, ω_{M_n} must be replaced by the angular velocity of the last inertia before backlash.

The paper is organized as follows. The section 2 represents the system. In the section 3, the vibration phenomenon is explained and the physical relation is presented. In the section 4, the two step approach to identify the backlash is explained and an algorithm is proposed. The simulation result are illustrated in the section 5.

2. STUDIED SYSTEM

The system consists of a driving motor and a load coupled to it through a gear and a shaft.The motor drive system with torsional load is shown in figure 2. The linear parameters are: J_M and J_L, inertia of the motor and of the load, f_M and f_L, viscous friction of the motor and of the load, f_{sh} and k_{sh}, coefficients of viscous damping and of elasticity of the shaft, respectively. The vector of linear parameters, P, is defined as:
$P = [J_M,\ J_L,\ f_M,\ f_L,\ f_{sh},\ k_{sh}]$. The only non linear parameter is the backlash amplitude, θ. The backlash is described by a dead zone model:

$$DZ_\theta(x) = \begin{cases} x-\theta & x \geq \theta \\ x+\theta & x \leq -\theta \\ 0 & |x| < \theta \end{cases} \tag{1}$$

The system can be represented by the following equations:

$$\begin{aligned} & J_M.\ddot{\phi}_M + f_M.\dot{\phi}_M = C_M - C_{sh}(\phi_d) \\ & J_L.\ddot{\phi}_L + f_L.\dot{\phi}_L = C_{sh}(\phi_d) \\ & C_{sh}(\phi_d) = \begin{cases} k_{sh}.(\phi_d-\theta) + f_{sh}(\dot{\phi}_d) & \phi_d \geq \theta \\ k_{sh}.(\phi_d+\theta) + f_{sh}(\dot{\phi}_d) & \phi_d \leq -\theta \\ 0 & |\phi_d| < \theta \end{cases} \\ & \phi_d = \phi_M - \phi_L \end{aligned} \tag{2}$$

where, $\dot{\phi}_M$ and $\dot{\phi}_L$, represent the motor and the load speed (the notations ω_M and ω_L are also used for them) and ϕ_d is the difference between the angles of the motor and of the load. The block diagram of the system is also shown in figure 2, where $g_1(s) = \frac{1}{J_M.s+f_M}$ and $g_2(s) = \frac{1}{J_L.s+f_L}$.

3. PHYSICAL RELATION

The torque, C_{sh}, has simultaneously the role of the exciting torque for $\omega_L = \dot{\phi}_L$ and of the resistant torque for $\omega_M = \dot{\phi}_M$. Each time the backlash input, $u_j = \phi_d$, enters or gets out of the dead zone, $[-\theta,\ \theta]$, corresponds to a commutation instant, t_{c_i} (see figure 4). It can be shown that for this system the amplitude of the oscillations of ω_M and ω_L decreases and the system is asymptotically stable (Appendix A).

The following relation exists between the back-

[2] $\omega_{M_n} = \omega_M + n_M$, $\omega_{L_n} = \omega_L + n_L$, where n_M and n_L represent the measurement noise.

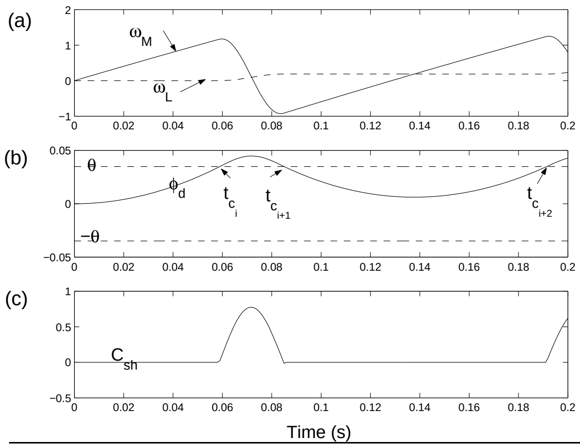

Fig. 4. Zoom on the step responses of system (around $t = 0$) , (a) the motor and the load speeds, ω_M and ω_L, (b) the backlash input ,$u_j = \phi_d$, and dead zone, $[-\theta, \theta]$, (c) the load torque, C_{sh}.

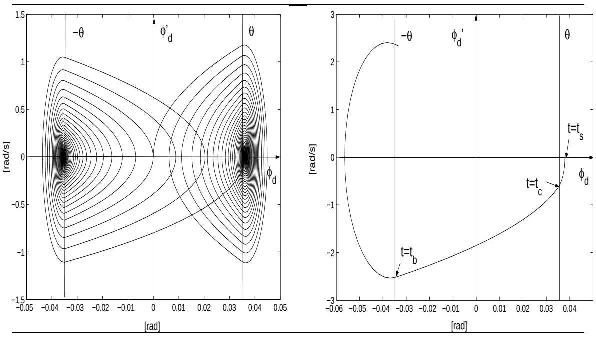

Fig. 5. Right : the phase diagram of system between $t = 0$ and $t = 2.t_s$, when $C_M(t) = 0.3 \times a_{max}.(1(t) - 1(t - t_s))$. Left: the same phase diagram around $t = t_s$. The commutation instants t_c and t_b are also shown.

lash amplitude, θ, the angular speeds ω_M and ω_L, the angular difference at $t = 0$, $\phi_d(0)$ and the vibration instant t_{c_i}:

$$\begin{aligned}\theta = |\phi_d(t_{c_i})| &= |\int_0^{t_{c_i}} \dot{\phi}_d.dt + \phi_d(0)| \\ &= |\int_0^{t_{c_i}} (\omega_M - \omega_L).dt + \phi_d(0)|\end{aligned} \quad (3)$$

It is shown that to eliminate $\phi_d(0)$, one can use the following relation (Appendix B):

$$\begin{cases} \theta = -0.5 \int_{t_c}^{t_b} (\omega_M(C_M(t)) - \omega_L(C_M(t))).dt \\ C_M(t) = a_1.1(t) - a_1.1(t - t_s) \end{cases} \quad (4)$$

In this relation, $t = t_s$ is an instant at which the system , with step input $C_M(t) = a.1(t)$, becomes stable, $t = t_c$ and $t = t_b$ are the two first vibration instants after $t = t_s$ (see also figure 5).

4. BACKLASH IDENTIFICATION

The backlash amplitude, θ, may be estimated by minimizing a least square criterion, evaluated for one of the system outputs (for example ω_M). If the domain of variations of θ is considered to be between θ_1 and θ_2, one can write:

$$\hat{\theta} = \arg \min_{\theta \in [\theta_1, \theta_2]} \sum_{t=t_i}^{t_j} (\omega_M(t) - \hat{\omega}_M(t, \theta))^2 \quad (5)$$

where $\hat{\omega}_M$ is the output of the model and it is found by replacing the uncertain linear parameters $\hat{P}$ in the system equations (2).

4.1 Pre-estimation of backlash amplitude

To use the relation (4) for backlash pre-estimation, one must notice that, firstly, the identification signals are noisy measurements of the motor and of the load speeds, ω_{M_n} and ω_{L_n}, thus, two low pass filters must be used, and , secondly, the commutation instants t_c, t_b are unknown. Therefore, the final relation for pre-estimation of backlash amplitude, $\hat{\theta}_{ini}$, is:

$$\hat{\theta}_{ini} = -0.5 \int_{\hat{t}_c}^{\hat{t}_b} (\omega_{M_f}(t) - \omega_{L_f}(t)).dt \quad (6)$$

where, $\hat{t}_c$,$\hat{t}_b$ are the estimations of t_c, t_b and the index f represents the filtered signals.

Estimation of the commutation instants, t_c, t_b: During the interval $[t_c, t_b]$, backlash is active and $C_{sh} = 0$. From the system equations (2):

$$J_L \dot{\omega}_L + f_L \omega_L = 0 \quad (7)$$

which results in:

$$\omega_L(t) = \omega_L(t_i) e^{-\alpha(t - t_i)} \qquad t,\ t_i \in [t_c,\ t_b] \quad (8)$$

In this relation, $\alpha = f_L/J_L$ is the time constant of the load. The exact values of t_c and t_b are noted by t_c^* and t_b^*. Three variables δt_1, δt_2 and δt_3 are selected to satisfy the following inequalities:

$$\begin{gathered} t_s < t_c^* < t_s + \delta t_1 < t_b^* \\ t_s + \delta t_2 < t_b^* < t_s + \delta t_3 < t_a \end{gathered}$$

where, t_a is the estimation of the third commutaion instant after t_s (figure 6). One defines the two following functions:

$$F_1(t, t_c) \equiv \begin{cases} \omega_L(t_c) & t_s \le t < t_c \\ \omega_L(t_c) e^{-\alpha(t - t_c)} & t_c \le t \le t_s + \delta t_1 \end{cases}$$

$$F_2(t, t_b) \equiv \begin{cases} \omega_L(t_b) e^{-\alpha(t - t_b)} & t_s + \delta t_2 \le t \le t_b \\ \omega_L(t_b) & t_b \le t \le t_s + \delta t_3 \end{cases}$$

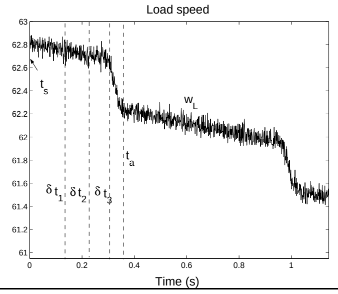

Fig. 6. The time intervals used for estimation of the commutation instants.

It is shown that the commutation instants can be estimated using the following relations (Appendix C):

$$\begin{aligned}
\hat{t}_c &= argmin(W_c(t_c))_{t_c \in]t_s, t_s+\delta t_1[} \\
W_c(t_c) &= \sum_{t=t_s}^{t_s+\delta t_1} (\omega_L(t) - F_1(t,t_c))^2 \\
\hat{t}_b &= argmin(W_b(t_b))_{t_b \in [t_s+\delta t_2, t_s+\delta t_3[} \\
W_b(t_b) &= \sum_{t=t_s+\delta t_2}^{t_s+\delta t_3} (\omega_L(t) - F_2(t,t_b))^2
\end{aligned} \quad (9)$$

These relations can be finally used after replacing the load speed, ω_L, by its noisy measurement, ω_{L_n}, and the time constant of rotation of the load $\alpha = \frac{f_L}{J_L}$ by its estimation $\hat{\alpha} = \frac{\hat{f}_L}{\hat{J}_L}$.

Filtering the measurement signals:

After estimating the commutation instants, $\hat{t}_c$ and $\hat{t}_b$, the frequency spectra of $\omega_{L_n}(t)$ and of $\omega_{M_n}(t)$, $\hat{t}_c < t < \hat{t}_b$ are traced and the cut off frequencies of FIR filters are found.

4.2 Final estimation of Backlash amplitude

The final estimation is performed using the pre-estimation $\hat{\theta}_{ini}$:

$$\begin{cases}
\hat{\theta} = \arg \min\limits_{\theta \in [\theta_1, \theta_2]} (W_1(\theta)) \\
W_1(\theta) = \sum\limits_{t=t_s}^{t_s+\delta t_4} (\omega_{M_n}(t, P^*) - \hat{\omega}_M(t, \hat{P}, \theta))^2 \\
C_M(t) = a_2.1(t) - a_2.1(t - t_s) \\
\theta_1 = \hat{\theta}_{ini}(1 - n_\theta) \\
\theta_2 = \hat{\theta}_{ini}(1 + n_\theta) \quad 0 < n_\theta < 1
\end{cases}$$

Comparing with the relation (5), the variables, t_i, t_j, and the input, $C_M(t)$, are chosen in the following manner:

1) $t_i = t_s$, to eliminate the dependence on the initial condition of the system, $\phi_d(0)$ (see Appendix B).

2) To reduce the influence of uncertain linear parameters, $\hat{P}$ in W_1:

- The identification time, $\delta_{t_4} = t_j - t_i$, must be small.
- The input, C_M, must not excite too much the linear dynamics but must excite the backlash , $C_M(t) = a.1(t) - a.1(t - t_s)$ (Appendix B).

Remark: The maximum of the applied step input, a_{max}, is limited because of the maximum allowable value of the motor speed $\omega_{M_{max}}$: $a_{max} = \frac{\omega_{M_{max}}}{gs}$, where g_s is the steady-state gain of the system.

4.3 Identification algorithme

1) Find the steady-state gain, g_s, and the stability time, t_s, using a step response of the system.
2) Compute the maximum allowable amplitude of the step inputs, a_{max}.
3) Apply : $C_M(t) = a_1.1(t) - a_1.1(t - t_s)$.
4) Estimate the commutation instances, t_c et t_b.
5) Filter the measurement signals, ω_{M_n} et ω_{L_n}.
6) Pre-estimate the backlash amplitude, $\hat{\theta_{ini}}$.
7)Apply : $C_M(t) = a_2.1(t) - a_2.1(t - t_s)$.
8)Estimate the backlash amplitude, $\hat{\theta}$.

5. SIMULATION RESULTS

The block diagram of the system, presented in figure 2, is simulated using:
$J_M = 4.88 \times 10^{-3}$, $J_L = 6.8 \times 10^{-2}$ $[Kg.m^2]$
$k_{sh} = 78$ $[\frac{N.m}{rad}]$, $f_M = f_L = 0.5 \times 10^{-2}$ $[\frac{N.m.s}{rad}]$
$f_{sh} = 1.575 \times 10^{-2}$ $[\frac{N.m.s}{rad}]$
$\theta = 2^0 \equiv 3.49 \times 10^{-2}$ rad, $T_s = 1$ ms
$SNR_{\omega_L} = 6.43$ db, $SNR_{\omega_M} = 17.4$ db (1^{st} step)
$SNR_{\omega_L} = 35.6$ db, $SNR_{\omega_M} = 35.7$ db (2^{nd} step).
At the second step of identification, when the estimated angular velocity of motor, $\hat{\omega}_M$, is used, the block diagram is simulated using the estimations of linear parameters, $\hat{P}$, and the pre-estimation of θ found in the first step, $\hat{\theta}_{ini}$. It is assumed that the estimations, $\hat{P}$, are random variables whose means are at the exact values , P^* (the values presented above). This assumption is applied using the formula $\hat{P} = P^*.(1 - Err_p.X)$, where, X is the vector of random variables with uniform distribution between -1 and 1, centered and of variance 1. Using the parameter Err_p, the maximum of the estimation error $P^* - \hat{P}$ can be modified. Evidently, the larger Err_p is the more erroneous the estimation $\hat{P}$ is.
For this system, $g_s = 100$ and t_s is chosen as the rise time $t_s = 40$ s. Supposing $\omega_{M_{max}} = 157$ $\frac{rad}{s}$, maximum allowable amplitude of step input is $a_{max} = 1.57$ $N.m$. In the identification algorithm, $a_1 = 0.1 \times a_{max}$, $a_2 = 0.6 \times a_{max}$, $n_\theta = 0.1$ and $\delta t4 = 1$ s. The cut off frequencies of the two FIR filters are 50 $\frac{rad}{s}$ and 20 $\frac{rad}{s}$ for ω_{M_n} and ω_{L_n}, respectively.
Considering the different values for the parameter Err_p, table 1 presents the estimations of θ

found by the proposed identification algorithm, $\hat{\theta}_{Al}$ and table 2 presents the estimation found by the following non-linear regression model, which does not use pre-estimation, $\hat{\theta}_{Reg}$:

$$\begin{cases} \hat{\theta}_{Reg} = \arg \min\limits_{\theta \in \theta_{mat}} W_{Reg} \\ W_{Reg} = \sum\limits_{t=t_s}^{t_s+\delta t_4} (\omega_M(C_M(t)) - \hat{\omega}_M(C_M(t), \theta))^2 \\ C_M(t) = a_2.1(t) - a_2.1(t - t_s) \\ \theta_{mat} = [0, 15^o] \end{cases}$$

This criterion is independent of the initial condition of the system, $\phi_d(0)$ (see section 4.2). For each value of Err_p , 10 experiences are performed. The backlash estimations, $\hat{\theta}_{Al}$ and $\hat{\theta}_{Reg}$, and the standard deviations, δ_{Al} and δ_{Reg}, presented in these two tables, are found based on these experiences. As can be seen, the backlash amplitude can be estimated by the proposed algorithm more precisely and with less standard deviation than with non linear regression method. In addition, the proposed algorithm is more robust according to the maximum error in the linear parameter estimation, Err_p, because the algorithm uses the pre-estimation which is found independently of the linear parameter estimation.

Table 1. The backlash estimated by the proposed algorithm during 10 experiences ($\theta^* = 3.49 \times 10^{-2}$).

$Err_p\%$	10%	20%	30%	40%
1	3.25	2.79	3.4	3.4
2	3.4	3.25	3.4	2.9
3	3.4	2.8	3.4	3.4
4	2.79	3.25	3.25	3.4
5	3.4	3.4	3.4	2.8
6	2.94	3.1	3.4	2.9
7	3.25	3.4	2.8	3.4
8	3.4	3.4	3.4	2.8
9	2.79	2.8	3.4	2.9
10	3.4	2.8	2.8	3.1
$100 \times \hat{\theta}_{Reg}$	3.21	3.1	3.27	3.11
$\pm err\%$	8%	11%	6.3%	10.8%
$\delta\theta \times 10^3$	2.64	2.83	2.58	2.7

Table 2. The backlash estimated by the non-linear regression method during 10 experiences ($\theta^* = 3.49 \times 10^{-2}$).

$Err_p\%$	10%	20%	30%	40%
1	3.49	5.24	5.23	1.74
2	0	0	3.49	1.74
3	3.49	5.24	20.9	0
4	5.24	12.2	0	20.9
5	3.49	1.74	12.2	12.2
6	0	5.24	5.24	3.49
7	3.49	1.74	1.74	5.23
8	0	0	3.49	1.74
9	0	0	1.74	10.5
10	3.49	3.49	2.09	5.24
$100 \times \hat{\theta}_{Reg}$	2.27	4.1	7.5	6.28
$\pm err\%$	34.9%	-17.4%	-115%	-80%
$\delta\theta \times 10^2$	2.02	3.6	7.8	6.49

6. CONCLUSION

A new approach for estimating the backlash amplitude, characterized by dead zone model, is proposed. Identification is achieved in two main steps. In the first step, a pre-estimation of the backlash amplitude, is performed based on the existing physical relation between the amplitude of the backlash, instant of commutation , the motor and the load speeds. In the second step, a least square criterion based on the model of the motor speed (non-linear regression model) is minimized around the pre-estimation.
The comparison between the backlash identification made by this approach and an approach that doesn't use the pre-estimation illustrates that the first method is considerably advantageous.

REFERENCES

D'Azzo, J. J. and C. H. Houpls (1981). *Linear control system.* McGraw-Hill Inc.. United states of America.

Stein, J.L. and C.H. Wang (1998). Estimation of gear backlash: Theory and simulations. *Transaction of ASME* **120**, 74–82.

Tao, G. (1996). Adaptive control of systems with nonsmooth input and output nonlinearities. *IEEE transaction on automatic control* **41**, 1348–1352.

Tao, G. and P. V. Kokotovic (1993). Adaptive control of systems with backlash. *Automatica* pp. 323–335.

Tian, M. and G. Tao (1997). Adaptive dead-zone compensation for output-feedback canonical systems. *Int. J. Control* **67**, 791–812.

Woo, K.T., Li-Xin Wang, F.L Lewis and Z.X. Li (1998). A fuzzy system compensator for backlash. *Proceedings of the 1998 IEEE International Conference on Robotics and Automation* pp. 181–186.

7. APPENDIX A: PROOF OF THE ASYMPTOTIC STABILITY

According to the equations (2), the 3 vectors of the state variables can be defined:

$$\phi_d \geq \theta : \begin{cases} \dot{X}_1 = A_1.X_1 \\ X_1 = [\dot{\phi}_M,\ \dot{\phi}_L,\ \phi_M - \phi_L - \theta]^T \end{cases}$$

$$\phi_d \leq -\theta : \begin{cases} \dot{X}_2 = A_2.X_2 \\ X_2 = [\dot{\phi}_M,\ \dot{\phi}_L,\ \phi_M - \phi_L + \theta]^T \end{cases}$$

$$|\phi_d| \leq \theta : \begin{cases} \dot{X}_3 = A_3.X_3 \\ X_3 = [\dot{\phi}_M,\ \dot{\phi}_L]^T \end{cases}$$

Considering $det(A_i) \neq 0,\ i = 1,2,3$, the three stability points, $X_{s_i},\ i = 1,2,3$, can be found by assuming $\dot{X}_i = 0,\ i = 1,2,3$:

$X_{s_1} \to \dot{\phi}_M = \dot{\phi}_L = 0,\ \ \phi_M - \phi_L = \theta$

$X_{s_2} \to \dot{\phi}_M = \dot{\phi}_L = 0,\ \ \phi_M - \phi_L = -\theta$

$X_{s_3} \to \dot{\phi}_M = \dot{\phi}_L = 0$

The system is asymptotically stable in the vicinity of these equilibrium points because, three scaler functions $V_1(X_1)$, $V_2(X_2)$ and $V_3(X_3)$ defined below, satisfy the 4 conditions of the Liapunov asymptotic stability, (D'Azzo and Houpls, 1981):

$$V_1(X_1) = \frac{1}{2}.J_M.X_1^2(1) + \frac{1}{2}.J_L.X_1^2(2) + \frac{k_{sh}}{2}.(X_1(3) - \theta)^2$$

$$V_2(X_2) = \frac{1}{2}.J_M.X_2^2(1) + \frac{1}{2}.J_L.X_2^2(2) + \frac{k_{sh}}{2}.(X_2(3) + \theta)^2$$

$$V_3(X_3) = \frac{1}{2}.J_M.X_3^2(1) + \frac{1}{2}.J_L.X_3^2(2)$$

Proof:

1) $V_i(X_i)$ is continuous, its partial derivatives are continuous at X_{s_i}.

2) $V_i(X_i) > 0$ for $X_i \neq X_{s_i}$, $i = 1,\ 2,\ 3$.

3) $V_i(X_i) = 0$, $i = 1,\ 2,\ 3$.

4) $\dot{V}_i(X_i) < 0$ for $X_i \neq X_{i_s}$ $i = 1,\ 2,\ 3$:

$$\dot{V}_1(X_1) = -f_M.X_1^2(1) - f_L.X_1^2(4) - f_{sh}.(X_1(1) - X_1(2))^2 < 0$$

$$\dot{V}_2(X_2) = -f_M.X_2^2(1) - f_L.X_2^2(2) - f_{sh}.(X_2(1) - X_2(2))^2 < 0$$

$$\dot{V}_3(X_3) = -f_M.X_3^2(1) - f_L.X_3^2(2) < 0$$

8. APPENDIX B: PROOF OF THE RELATION (4)

if $\phi_d(t_{c_i}) = \theta$ and $\phi_d(t_{c_{i+1}}) = -\theta$, then:

$$-\theta = \phi_d(t_{c_{i+1}}) = \int_0^{t_{c_{i+1}}} \omega_d.dt + \phi_d(0)$$

$$= \int_0^{t_{c_i}} \omega_d.dt + \int_{t_{c_i}}^{t_{c_{i+1}}} \omega_d.dt + \phi_d(0)$$

$$= \phi_d(t_{c_i}) - \phi_d(0) + \int_{t_{c_i}}^{t_{c_{i+1}}} \omega_d.dt + \phi_d(0)$$

$$= \theta + \int_{t_{c_i}}^{t_{c_{i+1}}} \omega_d.dt \Rightarrow \theta = -0.5 \int_{t_{c_i}}^{t_{c_{i+1}}} \dot{\phi}_d.dt$$

The last relation is independent of the initial condition, $\phi_d(0)$. To realize the conditions $\phi_d(t_{c_i}) = \theta$ and $\phi_d(t_{c_{i+1}}) = -\theta$, the input C_M is chosen to be: $C_M(t) = a.1(t) + b.1(t - t_s)$, $\quad b < -a$. In this relation, t_s is the instant at which the step response of the system becomes stable (vibrations are finished). It can be easily shown that this input transfers the stability point:
$X_g(t_{s_1}) = [a.g_s,\ a.g_s,\ \frac{f_L}{k_{sh}}.a.g_s + \theta]^T$, which (in the phase diagram) is in the right side of the dead zone, to another stability point:
$X_g(t_{s_2}) = [(a+b).g_s,\ (a+b).g_s,\ \frac{f_L}{k_{sh}}.(a+b).g_s - \theta]^T$, which is in the left side of the dead zone: This transformation requires a passage in the dead zone and the conditions $\phi_d(t_{c_i}) = \theta$ and $\phi_d(t_{c_{i+1}}) = -\theta$ are fulfilled (see figure 5). For more simplicity, in the paper, the notations t_c and t_b are used instead of t_{c_i} and $t_{c_{i+1}}$.

9. APPENDIX C: PROOFS OF THE RELATION (9)

Here, the criterion $W_c(t_c)$ is treated. The same operation can be applied on the other criterion $W_b(t_b)$.

$$W_c(t_c) = \sum_{t=t_s}^{t_c} (\omega_L(t) - \omega_L(t_c))^2 + \sum_{t=t_c}^{t_s+\delta t_1} (\omega_L(t) - \omega_L(t_c)e^{-\alpha(t-t_c)})^2$$

for $t_c^- < t_c^*$:

$$W_c(t_c^-) = \sum_{t=t_s}^{t_c} (\omega_L(t) - \omega_L(t_c))^2 + \sum_{t=t_c}^{t_c^*} (\omega_L(t) - \omega_L(t_c)e^{-\alpha(t-t_c)})^2 + \sum_{t=t^*c}^{t_s+\delta t_1} (\omega_L(t^*c).e^{-\alpha(t-t^*c)} - \omega_L(t_c).e^{-\alpha(t-t_c)})^2$$
$$= Err_1(t_c^-) + Err_2(t_c^-) + Err_3(t_c^-)$$

for $t_c = t_c^*$:

$$W_c(t_c^*) = \sum_{t=t_s}^{t_c^*} (\omega_L(t) - \omega_L(t_c^*))^2 = Err_1(t_c^*)$$

and finally, for $t_c^+ > t_c^*$:

$$W_c(t_c^+) = \sum_{t=t_s}^{t_c} (\omega_L(t) - \omega_L(t_c))^2 + \sum_{t=t_c}^{t_s+\delta t_1} (\omega_L(t_c^*)e^{-\alpha(t-t_c^*)} - \omega_L(t_c)e^{-\alpha(t-t_c)})^2$$
$$= \sum_{t=t_s}^{t_c} (\omega_L(t) - \omega_L(t_c))^2 = Err_1(t_c^+)$$

In the relation $W_c(t_c^+)$, the second sum is zero, because $t_c^* < t_c < t_s + \delta t_1 < t_b^*$, thus, $\omega_L(t) = \omega_L(t_c^*)e^{-\alpha(t-t_c^*)} = \omega_L(t_c)e^{-\alpha(t-t_c)}$.
By comparing the three criteria, one concludes that:

- $W_c(t_c^-) > W_c(t_c^*)$ due to the terms: $Err_2(t_c^-)$ and $Err_3(t_c^-)$.
- $W_c(t_c^+) > W_c(t_c^*)$.

Therefore, $W_c(t_c)$ is minimized at $t_c = t_c^*$. By the same method, it can be shown that $W_b(t_b)$ is minimized at $t_b = t_b^*$.