# ROBUST FRICTION COMPENSATION BASED ON KARNOPP MODEL

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**Keywords:** Friction compensation, Karnopp model, Describing Function, Stick and Slip Friction.

## Abstract

An essential problem concerning friction compensation is over-compensation or under-compensation. In model-based compensation for friction, these events are due to over-estimation or under-estimation of the model parameters. While generally under estimation results in steady state errors, over estimation may produce oscillations and instability. In this paper, a simple servo-mechanism where the friction part is modeled by Karnopp model is treated. A controller able to cope with these problems is proposed. Using describing function analysis the new design is studied.

## 1 Introduction

Friction as a nonlinearity which is present in all machines with motor part, has received much attention. This is due to its many undesirable effects like power dissipation, oscillation and steady state errors. In [1] a detailed discussion on friction, its analysis tools and the different methods for its compensation are given and the compensation methods are classified in two main groups: model-based and non-model-based. In non-model-based approaches, no friction model is used and generally compensation is performed by changing the parameters of the controller, for example the gain in the stiff PD [2] and in the integral controller [9], or the pulse width in impulsive control [3].

In the model-based compensation [5], one uses a friction model so that it is possible to compensate for friction by applying a command equal to the predicted force or torque and opposite to the friction force or torque.

The model parameters can be identified off-line or on-line. However, the risk of over-compensation or under-compensation, due to bad parameters estimation, always exists. Thus, in the model-based compensation, the over (or under) friction compensation must always be considered. Generally, under-compensation results in steady state error while over-compensation produces the oscillation.

In this paper, it is supposed that the parameters of a good friction model are identified off-line using an exciting input, then, a robust controller design is presented. This design prevents the change in the desired step response due to the change in the friction parameters of system. Canudas in [6] uses Describing Function (DF) to analysis the same problem. This paper is inspired from [6] but the Karnopp model is used to characterize friction. Karnopp model allows to explain the discontinuity of static to Coulomb friction force at the neighborhood boundary of zero velocity [7]. In this case, the DF of the both of the friction model and its compensator is a complex imaginary function of limit cycle amplitude and frequency [1].

In this complicated case, one can use a gain $g$, multiplying the predicted friction before applying to the system. This gain must be reduced when the effects of the over-compensation (oscillations) appear and increased when the steady state error due to under-compensation appears. As an exact relation between this gain, $g$, and the compensation error can not be found generally, the system needs a permanent supervision.

The robust design contains a parameter $k$ which can be adjusted according to the case of the maximum over-compensation and then remains constant. In addition, the values of the compensator parameters are chosen as the maximum values that the model parameters can have. Thus, friction is always over compensated. However, it is shown that in this approach, the step response of the system does not change significantly even if the true friction parameters change in a large domain. Also, limit cycle amplitude is negligible.

The paper is organized as follows: in section 2 , Karnopp model and its proposed compensator are presented. Section 3 illustrates two different designs to cope with over-compensation and under-compensation. The simulation results are presented in section 4. The DF computation of both of the Karnopp model and its compensator is done in Appendix.

## 2 Karnopp model and its compensator

Unlike many friction models that have velocity as input and predicted friction as output , the input and output of Karnopp model are respectively the applied force and the predicted velocity. parameters are: $F_c$ (Coulomb friction), $F_s$ (maximum stiction friction), $dv$ (limited velocity in stick-slip region) and $m$ (the mass of the moving part). The other symbols are: $F_e(t)$ ( applied force), $F_f(t)$ (friction force) and $\dot{y}(t)$ ( velocity).

When $\dot{y}(t)$ enters in the interval $[-dv, dv]$, the model output

[1] in [6] it has been shown that this DF always remains on the real axis.

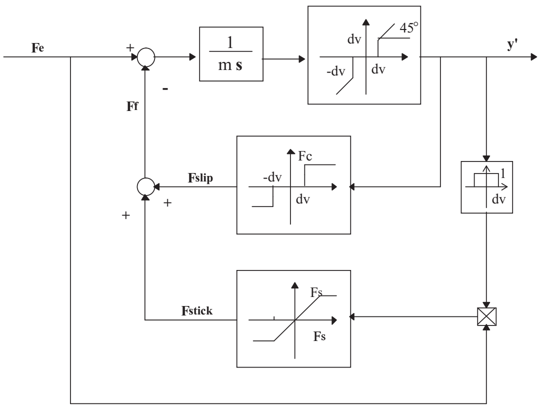


Figure 1: Block diagram of Karnopp model

is $\dot{y}(t) = 0$, the stick period begins and continues until the instant $t_1$, when a slip period begins. The instant $t_1$ satisfies the following equality:

$$\frac{1}{m}\int_{t_0}^{t_1}(F_e(t) - F_s.sign(F_e(t)))dt = 2.dv.sign(F_e(t)) \quad (1)$$

In this relation, $t_0$ is either the first instant when $F_e(t)$ is equal to $F_s$ and $\dot{F}_e(t_0) > 0$ or the first instant when $F_e(t)$ is equal to $-F_s$ and $\dot{F}_e(t_0) < 0$. $t_1$ represents the beginning of the slip period. During the slip period ($t > t_1$), $\dot{y}(t)$ is the solution of the following differential equation:

$$m.\ddot{y}(t) + F_c.sign(\dot{y}(t)) = F_e(t), \;\; \dot{y}(t_1) = dv.sign(\dot{y}(t)) \quad (2)$$

However as soon as the solution of this differential equation re-enters in the limited boundary at $t = t_2$ (*i.e.* $|\dot{y}(t_2)| \leq dv$ ), the slip period finishes. Note that according to the above discussion and figure (1), $(t_1 < t < t_2) \equiv (|\dot{y}(t)| > dv)$.
In short the estimated friction, $\hat{F}_f(t)$, will be:

- $F_e(t)$ , if $|F_e(t)| < \hat{F}_s$ and $|\dot{y}(t)| \leq \hat{dv}$,
- $\hat{F}_s.sign(F_e(t))$, if $|F_e(t)| \geq \hat{F}_s$ and $|\dot{y}(t)| \leq \hat{dv}$,
- $\hat{F}_c.sign(\dot{y}(t))$ , if $|\dot{y}(t)| > \hat{dv}$.

$\hat{F}_c, \;\hat{F}_s$ and $\hat{dv}$ must be found using an identification process as in [8].
Always friction compensator is considered as a block with velocity $\dot{y}(t)$ as input and estimated friction $F_{comp} = \hat{F}_f$ as output. During the slip periods ($|\dot{y}(t)| > \hat{dv}$), friction can be compensated by $F_{comp} = \hat{F}_c.sign(\dot{y}(t))$. However, during stick periods, $|\dot{y}(t)| \leq \hat{dv}$, friction compensation depends upon $F_e(t)$. To avoid precise measuring of $F_e(t)$, it is supposed that during stick periods $F_{comp} = \hat{F}_s.sign(F_e(t))$. This compensation increases $F_e(t)$ in the correct direction ($|F_e(t) + \hat{F}_s.sign(F_e(t))| \geq \hat{F}_s$), so that the stiction finishes very quickly. We propose the following compensator:

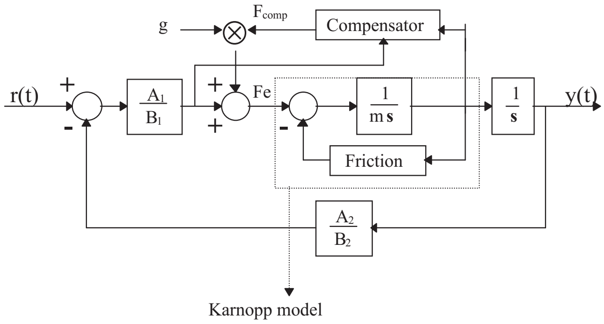


Figure 2: Block diagram of a simple servo-mechanism, the controllers and friction compensator.

$$F_{comp}(t) = \begin{cases} \hat{F}_c.sign(\dot{y}(t)) & |\dot{y}(t)| > \hat{dv} \\ \hat{F}_s.sign(F_e(t)) & |\dot{y}(t)| \leq \hat{dv} \end{cases} \quad (3)$$

To study the influence of incorrect compensation, three cases are considered:

- exact compensation $\hat{F}_c = F_c, \; \hat{F}_s = F_s$ and $\hat{dv} = dv$,
- over-compensation $\hat{F}_c = F_c(1 + \%n_c), \;\; \hat{F}_s = F_s(1 + \%n_s)$, and $\hat{dv} = dv$.
- under-compensation $\hat{F}_c = F_c(1 - \%n_c), \;\; \hat{F}_s = F_s(1 - \%n_s)$, and $\hat{dv} = dv$.

In these relations $\%n_c$ and $\%n_s$ present the maximum deviation in $F_c$ and $F_s$.

# 3 Two designs to cope with incorrect compensation

Figure(2) shows a simple servo-mechanism with two controllers $C_1 = \frac{A_1}{B_1}$, $C_2 = \frac{A_2}{B_2}$. Friction which is modeled by Karnopp model and the compensator are also marked. Gain $g$ must be regulated or constant ($g = 1$), corresponding to two designs that are discussed.
The controllers $C_1$ and $C_2$ must be designed to accomplish the desired closed loop transfer function :

$$H(s) = \frac{(\omega_n)^2}{s^2 + 2.\xi.\omega_n.s + (\omega_n)^2} \quad (4)$$

To find the controllers, always it is assumed that compensation is perfect (or friction does not exist).
To examine the robustness, DF analysis is used to show how limit cycle amplitude $a$ is changed if friction is over-compensated. Transfer function of linear part is named by $G(j\omega)$ and DF of non-linear part by $N(a, \omega)$. Frequency $\omega$ and amplitude $a$ of limit cycles must be hold in $G(j\omega).N(a, \omega) = -1$. Also, the limit cycles specifications can be estimated by

founding the intersections of the Nyquist diagrams of $G(j\omega)$ and $N(a, \omega)$.
For system presented in figure (2):

$$G(j\omega) = \frac{A_1.A_2}{j.\omega.B_1.B_2} \tag{5}$$

$$N(a, \omega) = N_{friction}(a, \omega) - g.N_{compensator}(a, \omega) \tag{6}$$

### 3.1 First design: design with variable compensator gain

Gain $g$ at the output of the compensator is tuned in order to eliminate the effects of the over-estimation or of the under-estimation.
The most simple controllers can be: $A_1 = m.\omega_n^2$, $A_2 = \frac{2.\xi}{\omega_n}s + 1$ and $B_1 = B_2 = 1$, which is a PD controller.
Df analysis
In this case,

$$G(j\omega) = 2.m.\omega_n.\xi - \frac{m.(\omega_n)^2.j}{\omega} \tag{7}$$

Obviously, it is not possible to modify the Nyquist diagram of $G(j\omega)$ to avoid the intersection with $N(a, \omega)$ without affecting the desired transfer function $H(s)$. Therefore, to avoid the intersections, gain $g$ in equation (6) must be regulated.

### 3.2 Second design: controller design

In this design the compensator gain $g$ is constant $g = 1$. The final objective in the new design is to reduce oscillation amplitude by modifying $G(j\omega)$ and in the same time, do not influencing $H(s)$.
It is supposed that: $H_1(s) = \frac{Y(s)}{R(s)} = \frac{A_1.B_2}{m.s^2.B_1.B_2 + A_1.A_2} = F(k, s).H(s)$. The proposed solution is:

$$A_1 = Ks, \;\; B_1 = s^2 + 2\xi\omega_n.s + \omega_n^2 \tag{8}$$

$$A_2 = b.(s^2 + 2\xi\omega_n.s + \omega_n^2), \;\; B_2 = 1 \tag{9}$$

This results in:

$$H_1(s) = \frac{k}{(ms + kb).\omega_n^2}.H(s) \tag{10}$$

Evidently the pole at $s = \frac{-kb}{ms}$ must be stable, thus $kb > 0$. Moreover, in order to obtain a steady state gain equal to one, $b$ has to satisfy $b = \omega_n^{-2}$, hence $k > 0$.
$H_1(s)$ can be extended as: $H_1(s) = \frac{A}{ms+kb} + \frac{Bs+C}{s^2+2\xi\omega_n.s+\omega_n^2}$ where $A = \frac{m^2}{b(kb-2m\xi\omega_n)}$, $B = \frac{m}{b(2m\xi\omega_n - kb)}$ and $C = b^{-1}$. The coefficients $A$ and $B$ decrease with increasing $k$ then the transient response improves. To find the realization of this new controller, $Y(s)$ can be rewritten as:
$Y(s) = \frac{k}{ms+b}.R_1(s)$ where $R_1(s) = \frac{R(s)}{s^2+2\xi\omega_n.s+\omega_n^2}$.
The new controller consists of a filter on the position reference and a derivation action on the filtered position error as can be seen in figure (3).
Note that in order that the position ($y(t)$) is similar to the reference ($r_1(t)$), the relation $s = \frac{-k.b}{m} \ll -1$ must be satisfied or

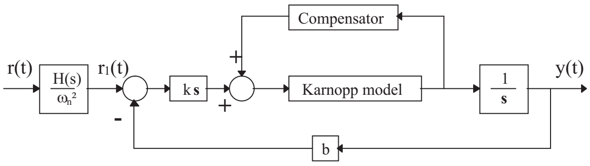

Figure 3: Block diagram of the second design.

in the other way, $k$ must be large.
DF analysis
In this design:

$$G(j\omega) = k.b = k.\omega_n{}^{-2} \tag{11}$$

Considering $N(a, \omega) = Re(N(a, \omega)) + j.Im(N(a, \omega))$, the limit cycle condition $G(j\omega).N(a, \omega) = -1$ will be:

$$k.\omega_n^{-2}.(Re(N(a, \omega)) + j.Im(N(a, \omega))) = -1 \tag{12}$$

In this case, each limit cycle $i$ corresponds to $Im(N(a_i, \omega_i)) = 0$ and $Re(N(a_i, \omega_i)) = -\frac{\omega_n^2}{k}$. Knowing that DF analysis implies some approximation, one can expect that choosing $k \gg 1$, both real and imaginary parts, must have small values at limit cycle $i$ which implies that the amplitude of oscillation ($a_i$) is small. This phenomena can be explained as following:
In the case of over-compensation, $F_e(t) = u(t) + F_s.(1 + n\%)sign(u)$ (see figure (2) overcomes $F_s$ quickly, hence, $\omega.t_1$ is very small. Assuming $\omega.t_1 = 0$ and ignoring the small valued $dv$, it can be shown that (see Appendix):

$$\omega.t_2 = \frac{a}{\tilde{F}_c}(1 - cos(\omega.t_2)) \tag{13}$$

In addition:

$$\begin{aligned} Im(N(a, \omega)) &= \tfrac{-1}{4\pi m}(\omega t_2) - \tfrac{1}{8\pi m}(sin(2\omega t_2) + \tfrac{2}{m\pi}sin(\omega t_2) \\ &\quad + \tfrac{2\tilde{F}_c}{m\pi a}(\omega t_2 sin(\omega t_2) - 1 + cos(\omega t_2)) \approx 0 \end{aligned} \tag{14}$$

$$\begin{aligned} Re(N(a, \omega)) &= -\tfrac{1}{2\pi m}(cos(2\omega t_2) - 1) - \tfrac{2}{m\pi}(1 - cos(\omega t_2)) \\ &\quad (-2 + cos(\omega t_2)) \approx 0 \end{aligned} \tag{15}$$

One understands that if $\omega.t_2$ becomes very small *i.e.* if slip period finishes more fast, the values of the real and imaginary parts will also be small. However, according to the equation(13), if $\tilde{F}_c$ is constant $\omega t_2$ diminution is equivalent with $a$ diminution or equivalent with better friction compensation.
To find a constant value for $k$ : three different cases are studied:

- $\tilde{F}_c = \tilde{F}_{c1}$ and $k = k_1$ which provide $a = a_1$ and $N(a, \omega) = N_1(a, \omega)$.
- $\tilde{F}_c = \tilde{F}_{c2}$, where $\tilde{F}_{c2} > \tilde{F}_{c1}$ and $k = k_1$ which provide $a = a_2$ and $N(a, \omega) = N_2(a, \omega)$.
- $\tilde{F}_c = \tilde{F}_{c3}$, where $\tilde{F}_{c3} = \tilde{F}_{c2}$, and $k = k_2$ where $k_2 > k_1$ which provide $a = a_3$ and $N(a, \omega) = N_3(a, \omega)$.

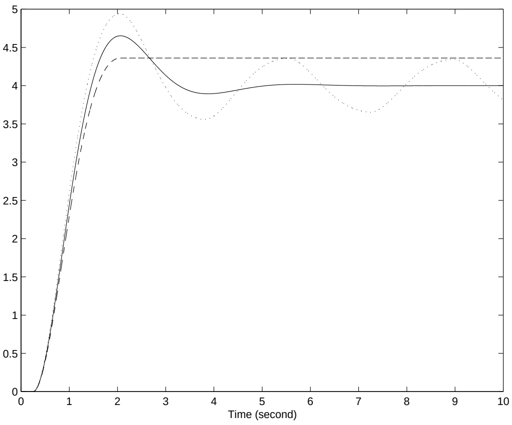


Figure 4: First design ($g = 1$), desired response (solid line), over friction compensation (dotted line) and under friction compensation (dashed line).

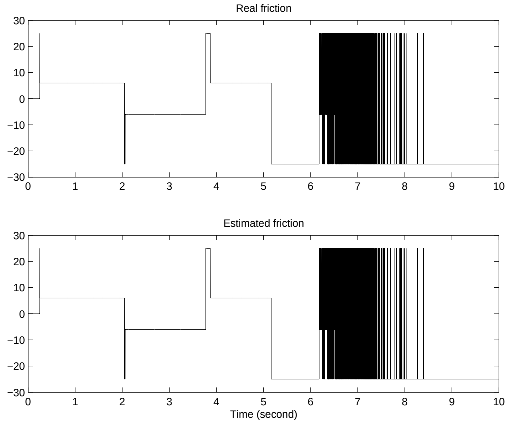


Figure 5: Real and estimated friction

There $\tilde{F}_{ci} = \hat{F}_{c_i} - F_c$. Suppose that in the case of $\tilde{F}_c = \tilde{F}_{c1}$, $k = k_1$ produces $a = a_1$ where $a_1$ is sufficiently small. If $\tilde{F}_{c2} = 2\tilde{F}_{c1}$, then $a_2 > a_1$. The experience shows that to obtain $a_3$ acceptable, $k_2 \approx 2k_1$ which is logically correct *i.e.* if $k_1$ is able to reduce the oscillation amplitude for $\tilde{F}_c = \tilde{F}_{c1}$, $2k_1$ will be able to do the same for $\tilde{F}_c = 2\tilde{F}_{c1}$ and this is may be due to the linearity of $N(.)$ in $\frac{\tilde{F}_c}{a}$.

Based on these discussions, the procedure is as follows:

- Identify Karnopp model parameters $\hat{F}_{c_i}, \hat{F}_{s_i}, \hat{dv}_i$.
- consider a maximum for friction parameters deviations $n\%$.
- choose $\hat{F}_c,\ \hat{F}_s$ and $\hat{dv}$ in equation (3) as $\hat{F}_{c_i}(1 + n\%)$ and $F_{s\,comp} = \hat{F}_{s_i}(1 + n\%)$ and $\hat{dv} = \hat{dv}_i$.
- find a value of $k$ which decreases considerably the oscillation amplitude. Name it $k_1$.
- choose $k = 2k_1$.

*Remark*: It can be noticed that only one tuning parameter $n$ is necessary for friction compensation. In addition, the same value of deviation for $F_c$ and $F_v$ has been used. However, as previously mentioned, $N(a, \omega)$ is more sensitive to the deviation in $F_c$ than to the deviation in $F_s$ and as we have checked in simulation, the deviation in $F_s$ can be much greater.

## 4 Simulation experiences

The desired step response characteristics are chosen as $\xi = .5,\ \omega_n = 2\ \frac{rad}{sec}$ and Karnopp model parameters as $F_s = 25\ N,\ F_c = 6\ N,\ dv = .02\ \frac{m}{s},\ m = 2\ kg$.
To examine the capability of the compensator presented in Eq.(3), the case of exact compensation ($\hat{F}_c = 6\ N,\ \hat{F}_s = 25\ N$ and $\hat{dv} = 0.02\ \frac{m}{s}$) are tested in the first design. Figure (5) illustrates the real friction ($F_f$ in figure (1)) and its estimation ( $\hat{F}_f$ found by equation(3)). As can be seen, they are nearly same.
First design
In figure (4), solid line, represents both of the desired step response and the actual one obtained using exact compensation ($\hat{F}_c = 6\ N,\ \hat{F}_s = 25\ N$ and $\hat{dv} = 0.02\ \frac{m}{s}$). The two curves are so close that the difference can not be distinguished in the figure.
To examine the robustness of this design, assuming that the compensator parameters are the same two cases will be consider:

- over-compensation $F_{c1} = 4,\ F_{s1} = 16.7$ and $dv = 0.02$.
- under-compensation $F_{c2} = 8,\ F_{s2} = 33.3$ and $dv = 0.02$

which is equivalent to $\pm 30\%$ deviation with respect to the values used in the compensator. Figure (4) presents the oscillations and the steady state error appeared due to applying this compensator for $F_{c1},\ F_{s1}$ and $F_{c2},\ F_{s2}$. To eliminate these effects, the gain $g$ must be decreased to $g = .66$ for the first case and increased to $g = 1.3$ for the second case. Thus the gain $g$ must always be tuned.
Second design
Considering the same desired values $\xi$ and $\omega_n$ and the same values for Karnopp model as in the last example the bloc diagram Fig.(3) is simulated. Figure 6, illustrates the performance of this new design when $k = 150$ and friction is exactly compensated ($\hat{F}_c = 6\ N,\ \hat{F}_s = 25\ N$ and $\hat{dv} = 0.02\ \frac{m}{s}$).
To find $k_1$, it is supposed that $n\% = 30$, then considering $\hat{F}_c = 8$ and $\hat{F}_s = 33.3$, $k$ is increased to obtain an acceptable oscillation. Figure (7( presents the system response with $k = 180,\ 250,\ 750$ and also the desired response. Accepting the response corresponding to $k_1 = 750$ ($a = .022$) , we consider the constant value of $k = 2 * k_1 = 1500$ as explained in the procedure. As figure (8) illustrates, the step response corresponding to $F_c = 4$ and $F_s = 16.7$, is always acceptable, even with nearly $2 \times 30\%$ over-compensation ($a = .04$).

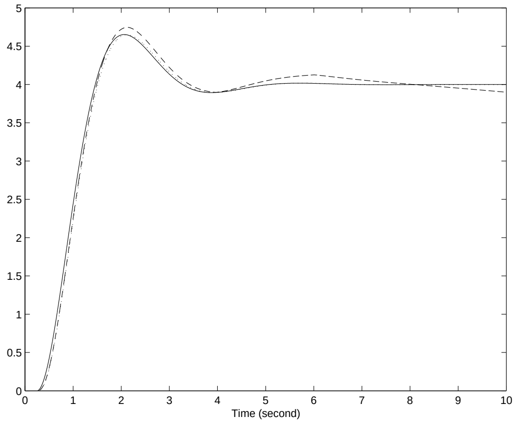


Figure 6: Second design ($k = 150$), desired step response (solid line), step response obtained by exact compensation (dotted line) and step response obtained by over-compensation (dotted line)

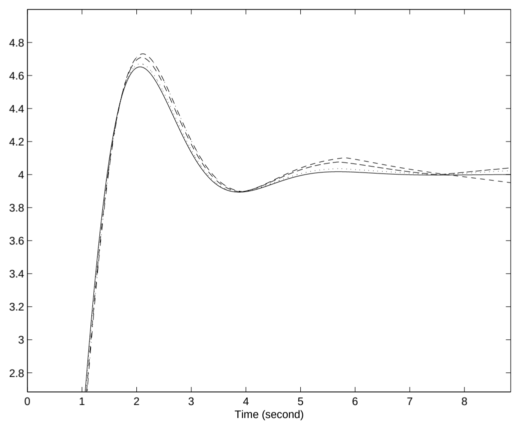


Figure 7: Second design, desired step response (solid line) and step responses obtained by $k = 180$ (dash-dot), $k = 250$ (dashed line) and $(k_1 = k) = 750$ (dotted line).

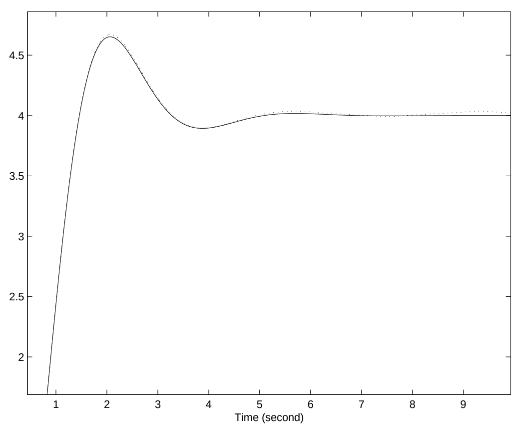


Figure 8: Second design, desired step response (solid line) and step response corresponding to $k = 2 * k_1 = 1500$, $60\%$ over-compensation (dotted line).

# 5 Conclusion

In this paper considering a simple servo mechanism, the problem of over-compensation or under-compensation when model-based compensation is used, is treated. Karnopp model is used to characterize friction. The main advantage of Karnopp model is its capability to shows stick-slip effect of friction. Two designs to cope with incorrect compensation are presented. The first one uses a gain $g$ for friction compensator. This gain must be increased to eliminate steady state error due to under-compensation or it must be increased to eliminate oscillation due to over-compensation. In the second design, we have chosen the values of the compensator parameters (Coulomb and stick friction parameters) equal to their maximum possible values. A controller design is proposed. It uses a gain $k$ that can tune the transfer function of the linear part in the complex plane without affecting the desired step response. This gain is adjusted according to the maximum possible deviation in the Coulomb and stick friction ($n\%$) which is the only parameter necessary for controller design. Being always in the case of over compensation, Describing Function (DF) analysis is used to show that oscillation amplitude is diminued.

# 6 Appendix

Describing Function (DF) of a nonlinearity is the complex ratio of the fundamental component of the nonlinear element by the input sinusoidal [4],*i.e.* $N(a,\omega) = \frac{M.e^{j(\omega.t+\phi)}}{a.e^{(j\omega.t)}} = \frac{M}{a}.e^{j\phi}$. Assume that Karnopp model together with its compensator (see figure (2) and section 2) is a nonlinearity with the applied force $F_e(t)$ as input and the velocity $\dot{y}(t)$ as output. We define $\tilde{F_s} = F_{s\,comp} - F_s$ and $\tilde{F_c} = F_{c\,comp} - F_c$. Note that $\dot{y}(t)$ is zero during the stick period and the influence of compensator is shortening of this period. Thus, any error in the estimation of $F_s$ influences the value of $t_1(\tilde{F_s})$. During the slip period , the compensator tries to compensate Coulomb friction. To compute the output $\dot{y}(t)$ for sinusoidal input, we use the differential equation (2), considering the compensator and replacing $F_e(t)$ by $a.sin(\omega.t)$ :

$$\begin{aligned} m.\ddot{y}(t) &= asin(\omega t) + (F_{c\,comp} - F_c)sign(\dot{y}(t)) \\ &= F_e(t) + \tilde{F_c}.sign(\dot{y}(t)) \\ \dot{y}(t_1) &= dv.sign(\dot{y}(t)) \end{aligned} \quad (16)$$

If $\dot{y}(t) > 0$, the solution is:

$$\dot{y}(t) = \frac{a}{m\omega}(cos(\omega t_1) - cos(\omega t)) + \frac{\tilde{F_c}}{m}(t - t_1) \quad (17)$$

Note that both the input and the output are periodic signals with period $T = \frac{2\pi}{\omega}$, then $\dot{y}(t + \frac{T}{2}) = -\dot{y}(t)$. Thus in the Fourier series of the output, even coefficients do not exist. Using the first odd coefficients, it is obtained that: $N(a) = \frac{a_1.cos(\theta)+b_1.sin(\theta)}{a.sin(\theta)} = \frac{a_1.j+b_1}{a}$. Replacing $a_1$ and $b_1$ from the Fourier series definition, we obtain: $N(a) = \frac{4.j}{a.T}.\int_0^{\frac{T}{2}} \dot{y}(t).e^{-j.\theta}.d\theta$. As Eq.(17) illustrates, $\dot{y}(t)$ is a function of $t = \frac{\theta}{\omega}$. Thus, the variable $\theta$ is replaced by $\omega.t$. In addition only the slip period is used for integration (during the stick period $\dot{y}(t) = 0$). Thus, $N(a,\omega)$ can be computed as:

$$N(a,\omega) = \frac{4.j.\omega}{a.T}.\int_{t_1}^{t_2} \dot{y}(t)e^{-j\omega.t}dt \tag{18}$$

The results are:

$$\begin{aligned} Im(N(a,\omega)) \quad &= \tfrac{-1}{\pi.m}(\omega.t_2 - \omega.t_1)(.25 + \tfrac{2.\tilde{F}_c}{a}(sin(\omega.t_2) - \\ &sin(\omega.t_1)) - \tfrac{1}{8\pi.m}(sin(2\omega.t_2) - sin(2\omega.t_1)) \\ &- \tfrac{2.\tilde{F}_c}{\pi.m.a}(cos(\omega.t_2) - cos(\omega.t_1)) \\ &+ \tfrac{2}{m.\pi}(sin(\omega.t_2) - sin(\omega.t_1))(cos(\omega.t_1) + \\ &\tfrac{\tilde{F}_c}{m.a}\omega.t_1) \end{aligned} \tag{19}$$

$$\begin{aligned} Re(N(a,\omega)) \quad &= \tfrac{-2.\tilde{F}_c}{\pi.m.a}(\omega.t_2 - \omega.t_1) \\ &(cos(\omega.t_2) - cos(\omega.t_1)) - \tfrac{1}{2\pi.m}(cos(2\omega.t_2) \\ &-cos(2\omega.t_1)) - \tfrac{2.\tilde{F}_c}{\pi.m.a}(sin(\omega.t_1) - sin(\omega.t_2)) \\ &+ \tfrac{2}{m.\pi}(cos(\omega.t_2) - cos(\omega.t_1))(cos(\omega.t_1) \\ &+ \tfrac{\tilde{F}_c}{m.a}\omega.t_1) \end{aligned} \tag{20}$$

## References


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